TWO-LAYER "SHALLOW WATER" MODEL

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INTRODUCTION

The mathematical model of "shallow water", i.e., the asymptotic theory of unsteady wave motions on the surface of a thin, ideal incompressible fluid layer in a transverse gravity field, is well known [1]. The agreement between the fundamental differential equations and the gasdynamics equations of isentropic motions of a polytropic gas with polytropy index two is especially remarkable. This theory was formulated in [2] on a strong asymptotic basis, a definite strict foundation was obtained in [3, 4], and in this sense can be considered complete. Meanwhile, oceanology problems result in the need to study the wave motions of stratified fluids, i.e., that incompressible fluid of variable density which is stratified in almost horizontal layers by a family of isochoric surfaces. Of special interest are such cases of stratification when the number of layers is finite, and the density is constant in each layer. The possibility of sliding of one layer over the other is hence allowed, i.e., the formation of contact discontinuities on the boundaries of layers of different density. Such motions are of interest in that because of the transfer of momentum from one layer to another, waves of considerable amplitude with comparatively slightly perturbed outer surface, the so-called internal waves, can be formed [5, 6]. The exact hydrodynamic theory of the formation and development of internal waves is quite difficult and barely advanced (one of the few exact results on stationary waves is obtained in [7]). Hence, the construction and investigation of the simplest models containing the basic singularities should be the first steps in a study of this phenomenon. One such model, linear theory, is already almost one hundred years old and has been studied well enough. It is impossible to say this about the nonlinear theory of nonstationary internal waves, in which the state of the art is related principally to the numerical solution of individual problems. There is clearly a deficit in analytical investigations in this area.

This paper is devoted to the derivation and preliminary analysis of three mathematical wave-motion models of a two-layer fluid in the asymptotic "shallow water" approximation. Only the case when the lighter fluid is above the heavier is examined here.

The first model describes motion with a free upper boundary and is an autonomous quasilinear homogeneous system of four first order differential equations. In principle, the singularity of this system is that it is of composite type in a definite domain of values of the depth of the layers and the flow velocities while outside this domain it is strictly hyperbolic. This fact needs a subsequent detailed analysis since the possible incorrectness of the natural Cauchy problem for wave motions is related thereto.

In the second model the upper boundary is a horizontal impermeable wall. Here the two-layer fluid moves in a horizontal tube, or, as is said in the text, "under a cover." This alters the boundary condition on the outer boundary substantially, and introduces a significant simplification in the model. Despite the fact that the method of simplifying problems by introducing an upper impermeable wall in papers on internal waves is used sufficiently extensively, the equations of two-layer "shallow water" that is "under a cover" have apparently not been encountered in previous literature. In this model, which consists of just two differential equations, the above-mentioned singularity appears in a more explicit form. The system obtained turns out to be simply a system of mixed elliptic–hyperbolic type. The question of the correctness of the Cauchy problem hence occurs especially acutely.

The third model is of more particular nature and is subject to the first two. It refers to the case of a small relative difference in the layer densities and low Froude numbers of the relative slip of one layer on the other. As a result of the specific asymptotic representation of the nature of this smallness, both the preceding models result in the very same system of two equations of mixed type. The attractiveness of this system is that the Riemann invariants are calculated explicitly, and the whole analytical structure of the model is thereby made transparent. The remarkable fact that the system obtained is exactly equivalent to the system of gasdynamics equations of a polytropic gas with the polytropy index 2 is hence established, where the solutions of the gasdynamics equations must also be considered even for negative values of the gas density.

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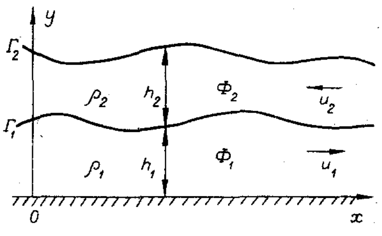

Fig. 1

It can be hoped that a further detailed investigation of these relatively simpler models will permit separation of the nonlinear wave motions of a two-layer fluid into the essential singularities in order to produce a basis for the development of an exact theory and to make suitable recommendations of an applied nature.

MODEL I

The plane-parallel unsteady motion of an incompressible fluid above a level horizontal bottom is considered in a gravity field (g = 1). It is assumed that the fluid consists of two immiscible layers of different density in each of which the motion is potential (Fig. 1). The pressure on the upper free boundary is considered zero.

Let Φ_i, h_i, ρ_i, Γ_i be the velocity potential, depth, density, and upper boundary of the i-th layer (i = 1, 2), respectively. If boundary conditions for the potentials $\varphi_{ij} = \Phi_i|\Gamma_j$ are introduced by the formulas (i, j = 1, 2; $i \geq j$), then the following system of equations is obtained from the kinematic and dynamic conditions on the boundaries Γ_i:

$$h_{1t} + \varphi_{11x}h_{1x} = (1 + h_{1x}^2)N_{11}, \quad h_{1t} + \varphi_{21x}h_{1x} = (1 + h_{1x}^2)N_{21}, \tag{1}$$
$$(h_1 + h_2)_t + \varphi_{22x}(h_1 + h_2)_x = (1 + (h_1 + h_2)_x^2)N_{22},$$
$$\rho_1\left(\varphi_{11t} + \frac{1}{2}\varphi_{11x}^2 + h_1\right) - \rho_2\left(\varphi_{21t} + \frac{1}{2}\varphi_{21x}^2 + h_1\right) = \frac{1}{2}(1 + h_{1x}^2)(\rho_1 N_{11}^2 - \rho_2 N_{21}^2),$$
$$\varphi_{22t} + \frac{1}{2}\varphi_{22x}^2 + h_1 + h_2 = \frac{1}{2}(1 + (h_1 + h_2)_x^2)N_{22}^2,$$

where $N_{ij} = \Phi_{iy}|\Gamma_j$ (i, j = 1, 2), and the subscripts t, x, y denote the partial derivatives with respect to the corresponding arguments.

Modeling is performed in the manner of the "shallow water" theory, in which the change of variable is made (the new quantities on the right of the → symbol are henceforth called model quantities)

$$y \to \varepsilon y, \quad h_i \to \varepsilon h_i, \quad \Phi_i \to \varepsilon^{1/2}\Phi_i, \quad \varphi_{ij} \to \varepsilon^{1/2}\varphi_{ij}, \quad t \to \varepsilon^{-1/2}t \tag{2}$$

and the subsequent formal passage to the limit as $\varepsilon \to 0$ is based on the lemma following below. Model quantities from (2) take part, where $N_{ij} \to \varepsilon^{-1/2}N_{ij}$, a suitable smoothness of the functions h_i, φ_{ij} is assumed, and the condition of nonpenetration at $y = 0$ is taken into account.

<u>FUNDAMENTAL LEMMA.</u> The limit relations

$$\lim_{\varepsilon\to 0}(\varphi_{21} - \varphi_{22}) = 0, \quad \lim_{\varepsilon\to 0}\varepsilon^{-2}N_{11} = -h_1\varphi_{11xx}, \tag{3}$$
$$\lim_{\varepsilon\to 0}\varepsilon^{-2}N_{21} = h_{1x}\varphi_{21x} - (h_1\varphi_{11x})_x,$$
$$\lim_{\varepsilon\to 0}\varepsilon^{-2}N_{22} = h_{1x}\varphi_{21x} - h_2\varphi_{21xx} - (h_1\varphi_{11x})_x,$$

are valid for the model quantities (2), where the first allows differentiation with respect to t and to x.

The formal proof of this lemma is based on the representations of the model potentials

$$\Phi_i = \Phi_i^0 + \varepsilon^2\Phi_i^1 + O(\varepsilon^4),$$

whose components are sought under suitable boundary conditions after substitution into the equation

$$\varepsilon^2\Phi_{ixx} + \Phi_{iyy} = 0$$

and comparison of terms with identical powers of ε.

Taking (3) into account, (1) for the model quantities pass to the limit into the system of two-layer "shallow water" equations after slight manipulation (the dynamic relationships are differentiated with respect to x)

$$h_{1t} + u_1h_{1x} + h_1u_{1x} = 0, \quad h_{2t} + u_2h_{2x} + h_2u_{2x} = 0, \tag{4}$$

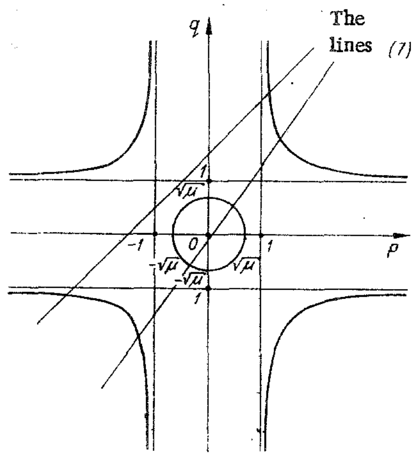

Fig. 2

$$u_{1t} + u_1u_{1x} + h_{1x} + \lambda h_{2x} = 0, \; u_{2t} + u_2u_{2x} + h_{1x} + h_{2x} = 0, \tag{4}$$

where $u_i = \varphi_{iix}$ (i = 1, 2); $\lambda = \rho_2/\rho_1$. The usual equations of single-layer "shallow water" are obtained from (4) for the quantities h_1 and u_1 if we set $\lambda = 0$ and discard the second and fourth equations. The first two equations of (4) express the differential "mass conservation" laws for each of the layers, and the last two express the differential "momentum conservation" laws for these layers. More exactly, we should speak of the exchange of momentum between layers in this latter case.

The properties of the wave motions of a two-layer fluid described by the system (4) are closely allied to its type. Their angular coefficient $k = dx/dt$ is introduced to analyze the characteristics, and the characteristic determinant D(k), a fourth degree polynomial in k, is formed by a known rule. A calculation yields

$$D(k) = ((u_1 - k)^2 - h_1)((u_2 - k)^2 - h_2) - \lambda h_1 h_2.$$

A graphic geometric representation of the arrangement of the roots of the equation $D(k) = 0$ is obtained if the quantities p, q are introduced by the relationships

$$u_1 - k = p\sqrt{h_1}, \; u_2 - k = q\sqrt{h_2}. \tag{5}$$

The equation $D(k) = 0$ takes the standard form

$$(p^2 - 1)(q^2 - 1) = \lambda, \tag{6}$$

and elimination of k from (5) yields the relationship

$$q = \sqrt{h_1/h_2}\cdot p + (u_2 - u_1)/\sqrt{h_2}. \tag{7}$$

On the (p, q) plane (6) describes a fourth order curve having four axes of symmetry, while (7) is a line with a positive angular coefficient $\sqrt{h_1/h_2}$ and an initial ordinate $(u_2 - u_1)/\sqrt{h_2}$. These lines are shown in Fig. 2, where $\mu = 1 - \lambda$ and it is assumed (as well as everywhere below) that $0 < \lambda < 1$.

It follows from Fig. 2 that the line (7) always has at least two points of intersection with the curve (6) for nonzero h_1, h_2, and a maximum of four such points. Since each point of intersection of the curve (6) with the line (7) yields a characteristic with the slope $k = u_1 - p\sqrt{h_1}$, then the following deduction is obtained. The system (4) is of mixed type: It is strictly hyperbolic in certain solutions (four real characteristics), and a system of composite type in some (two real and two imaginary characteristics); the passage from one to the other is obtained upon the merger of the two characteristics into one (corresponding to the tangency of the line (7) and the curve (6)).

By relying on this geometric representation of the characteristics, it can be noted that the composite type of the system (4) is realized in the domain of values of h_i, u_i of the form

$$\sqrt{h_1}f_1(h_2/h_1) < |u_2 - u_1| < \sqrt{h_1}f_2(h_2/h_1)$$

with certain specific functions f_i dependent only on λ. Hence, $f_1 \geq \sqrt{\mu}$, $f_2 \geq 1$. Therefore, strict hyperbolicity holds either for sufficiently small or for sufficiently large values of the Froude number defined relative to the velocity

$$f = |u_2 - u_1|/\sqrt{h_1}.$$

The factor of the existence of a domain of variables h_i, u_i in which the system (4) is of composite type impugns (but does not disprove!) the correctness of the Cauchy problem with arbitrary initial data at $t = 0$.

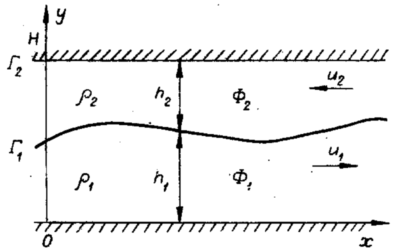

Fig. 3

PARTICULAR SOLUTIONS

A group analysis of the system (4) shows that it admits of a 5-parameter fundamental group generated by one-parameter transformation groups in t and in x, a Galilean transformation, and two independent extensions. This permits the investigation of the different particular solutions, which are sought by the reduction to systems of ordinary differential equations. For instance, simple waves are described by using the parameter k, which has the meaning of a slope of the characteristics of the straight-line family to the t axis, by means of the following system:

$$dh_1/dk = -\sqrt{h_1h_2}(q^2-1)A,\quad dh_2/dk = -\sqrt{h_1h_2}A,$$
$$du_1/dk = \sqrt{h_2}(q^2-1)pA,\quad du_2/dk = \sqrt{h_1}qA,$$

where

$$A = \frac{3}{2}\,\frac{p(q^2-1)\sqrt{h_2}+q(p^2-1)\sqrt{h_1}}{p^2(q^2-1)^2h_2+q^2(p^2-1)h_1},$$

and the quantities p and q are defined by (5) and are related by (6) and (7).

Still another family of particular solutions is described by using the assumption about the polynomial dependence of the desired h_i, u_i on the variable x. Namely, the system (4) has exact solutions of the form

$$u_i = \alpha_i(t)x + \beta_i(t),\ h_i = \gamma_i(t)x^2 + \delta_i(t)x + \varepsilon_i(t)\quad (i = 1,\ 2\ ),$$

where the functions α_i, ..., ε_i satisfy a definite normal system of ordinary first-order differential equations obtained by substituting these expressions into the system (4) and "splitting" it with respect to the variable x.

It is also useful to note that the system (4) allows being written in the form of the conservation laws

$$M_t + N_x = 0 \tag{8}$$

with functions M, N dependent on h_i, u_i. In all there are six, and only six, linearly independent conservation laws of the form (8). Four are seen directly from the writing of the system (4), where the function M equals one of the quantities h_i or u_i. Two additional conservation laws are true with the functions

$$M = 2h_1u_1 + 2\lambda h_2u_2,\quad N = 2h_1u_1^2 + 2\lambda h_2u_2^2 + h_1^2 + 2\lambda h_1h_2 + \lambda h_2^2$$

or

$$M = h_1u_1^2 + \lambda h_2u_2^2 + h_1^2 + 2\lambda h_1h_2 + \lambda h_2^2,$$
$$N = h_1u_1^3 + \lambda h_2u_2^3 + 2h_1^2u_1 + 2\lambda h_1h_2(u_1+u_2) + 2\lambda h_2^2u_2.$$

In the limit case $\lambda = 0$, the motion in the lower layer is determined by the known single-layer "shallow water" equations. The other limit case $\lambda = 1$ will be considered below.

MODEL II

The analogous motion of a two-layer fluid is described by perfectly different equations in the case when the upper layer is "under a cover" (Fig. 3). Here the boundary Γ_2 is a horizontal impermeable wall (the fluid is "fixed" from above and below). In this case the third and fifth equations drop out of the system (1) and the impermeability condition $\Phi_{2y}|\Gamma_2 = 0$ is added.

The very same modeling process (2) and the application of an analog of the fundamental lemma result in the following system of equations for the model quantities

$$\begin{gathered} h_1 + h_2 = H,\ h_{1t} + u_1h_{1x} + h_1u_{1x} = 0, \\ h_{2t} + u_2h_{2x} + h_2u_{2x} = 0, \\ u_{1t} + u_1u_{1x} + h_{1x} = \lambda(u_{2t} + u_2u_{2x} + h_{2x}). \end{gathered} \tag{9}$$

The "flow-rate integral" follows from the first three equations of (9):

$$u_1h_1 + u_2h_2 = a(t).$$

In the absence of sources there must be $a(t) = a =$ const. In a coordinate system moving uniformly with the velocity a/H (Galilean transformation), there will be $a = 0$. Hence, if we put $h_1 = h$, then the formulas

$$u_1 = u, \; u_2 = -[h/(H-h)]u$$

will be valid for the velocities of the layers. This permits the elimination of u_2 and h_2 from the last equation in (9), which results in a system of two-layer "shallow water" equations "under a cover":

$$h_t + uh_x + hu_x = 0,$$
$$(H + \mu h)u_t + \frac{H^2 - (2+\lambda)Hh + \mu h^2}{H-h} uu_x + \left(\mu(H-h) - \frac{\lambda H^2}{(H-h)^2}u^2\right)h_x = 0,$$

where, as before, $\mu = 1 - \lambda$. A more symmetric form of these equations is obtained if a "normalized" velocity v is introduced instead of u by the formula

$$u = [(H-h)/(H-\mu h)]v.$$

Then the preceding system is converted into the following:

$$h_t + \frac{H^2 - 2Hh + \mu h^2}{(H-\mu h)^2} vh_x + \frac{h(1-h)}{H-\mu h} v_x = 0,$$
$$v_t + \frac{H^2 - 2Hh + \mu h^2}{(H-\mu h)^2} vv_x + \left(\mu - \frac{\lambda H^2}{(H-\mu h)^3} v^2\right) h_x = 0. \qquad (10)$$

Evaluation of the characteristics of the system (10) yields the following expression for the angular coefficient $k = dx/dt$:

$$k = \frac{H^2 - 2Hh + \mu h^2}{(H-\mu h)^2} \pm \sqrt{\frac{h(H-h)}{H-\mu h}} \sqrt{\mu - \frac{\lambda H^2 v^2}{(H-\mu h)^3}}. \qquad (11)$$

It is seen from (11) that the system (10) is of mixed type. It is strictly hyperbolic in the strip $0 < h < H$ for those (v, h) for which

$$|v| < \sqrt{\mu/\lambda} \cdot H^{-1}(H - \mu h)^{3/2} \qquad (12)$$

while the system (10) is elliptic for values of (v, h) satisfying the opposite inequality.

Unfortunately, the differential equation of the characteristics in the (v, h) plane is not integrated explicitly here. The domain (12) and the corresponding network of characteristics obtained by a numerical computation are shown in Fig. 4.

It is useful to note that on the transition line where

$$v = \pm\sqrt{\mu/\lambda} \cdot H^{-1}(H - \mu h)^{3/2},$$

the characteristics in the (v, h) plane have a reentry point. Moreover they have a second-order tangent with the singular lines $h = 0$ and $h = H$.

Model II is remarkable for the fact that it yields an example of the evolutionary system of two differential equations which has an explicit physical meaning and is a system of mixed type. Study of the physically meaningful problems in model II is fraught with definite difficulties. In fact, the correct formulation of the Cauchy problem with initial data at $t = 0$ is typical for evolutionary systems on the one hand. On the other hand, the emergence of the quantities v, h in the domain of ellipticity of the system (10) should result in a Cauchy problem with the development of a Hadamard instability. The question of in what manner the model II handles the destructive tendencies which occur still remains open. The absence of any known analogs of this situation in hydrogasdynamics should be noted, where systems of mixed type have only been encountered in stationary problems until now.

The specific analytical difficulties in investigating the solutions of the system (10) are related to the fact that the Riemann invariants are not calculated here in an explicit final form. This circumstance suggested a study of the limit case $\mu \to 0$ in which, as it turns out, explicit analytical formulas are obtained for the Riemann invariants. However, if we simply set $\mu = 0$ in the system (10), by considering all the remaining quantities finite and nonzero, then it reverts into a system of elliptic type. More exactly, the elliptic type is obtained for any $v \neq 0$ while parabolic degeneration occurs for $v = 0$. This last remark shows that it is possible to expect to obtain a substantially new model if the simultaneous passages to the limit $\mu \to 0$ and $v \to 0$ are made.

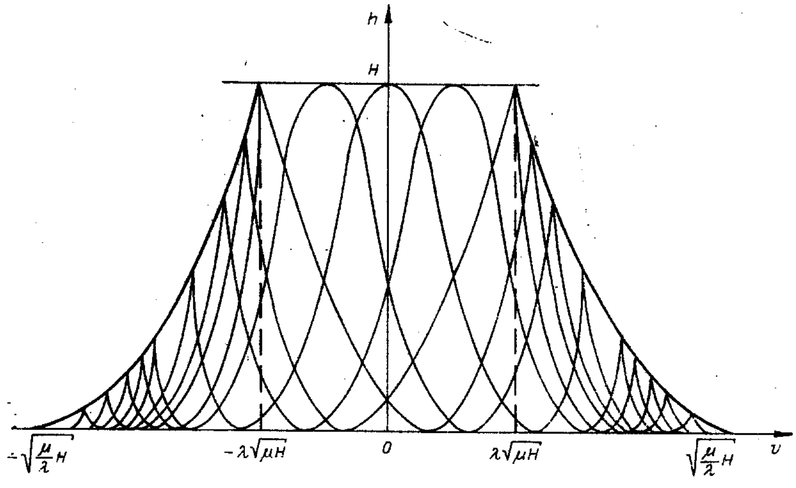

Fig. 4

MODEL III

To obtain a simpler model of mixed type, a new modeling with the small parameter μ independent of (2) is performed in model II:

$$h \to Hh,\ v \to \sqrt{\mu H}v,\ t \to t/\sqrt{\mu H}. \tag{13}$$

Substitution into (10) and the formal passage to the limit $\mu \to 0$ results in the following system of equations for the model quantities:

$$\begin{aligned} &h_t + (1 - 2h)vh_x + h(1 - h)v_x = 0, \\ &v_t + (1 - 2h)vv_x + (1 - v^2)h_x = 0. \end{aligned} \tag{14}$$

Henceforth, the hydrodynamic system described by (14) will be called "model III."

Before turning to its analysis, it is useful to note that the very same equations (14) are obtained if modeling of the kind (13) is applied to the system (4). Indeed, the first two equations of (4) hence remain invariant for the model quantities, while the last two take the form

$$\begin{aligned} &\mu(u_{1t} + u_1u_{1x} - h_{2x}) + (h_1 + h_2)_x = 0, \\ &\mu(u_{2t} + u_2u_{2x}) + (h_1 + h_2)_x = 0. \end{aligned} \tag{15}$$

In the limit as $\mu \to 0$ it is hence obtained that $h_1 + h_2 = b(t)$. But then the first two equations yield the "flow rate integral"

$$h_1u_1 + h_2u_2 = -b'(t)x + c(t).$$

The assumption about no sources, analogous to that made in deriving the model II, implies the necessary equalities $b(t) = b = \text{const}$ and $c(t) = c = \text{const}$. It is clear that we can make $b = 1$ because of a suitable choice of the constant H in a modeling of the kind (13). Hence, additional utilization of the Galilean transfer results in the relationships

$$h_1 + h_2 = 1,\ h_1u_1 + h_2u_2 = 0, \tag{16}$$

whereupon the first two equations in (4) reduce to one, for instance, the first of them. To obtain an additional equation (the second in the desired model system) it is necessary to form such an equation from (15) as would not contain higher approximations in μ; it is obtained by subtracting (15)

$$u_{1t} + u_1u_{1x} - h_{2x} - u_{2t} - u_2u_{2x} = 0.$$

It can now be verified that after eliminating h_2 and u_2 by using (16), the system (14) is obtained exactly for the quantities $h = h_1$ and $v = u_1/(1 - h)$.

An interesting and important question about the foundation of a modeling of the kind (13) in the parameter μ occurs in connection with model III. The nontriviality of this question is related to the difference in the behavior of the characteristics of the systems (14) and (10) on the line of parabolicity.

Thus, the model III is intended to describe wave motions on the interface of a two-layer fluid with a small positive relative difference in the densities $\mu = (\rho_1 - \rho_2)/\rho_1$ for both a layer "under a cover" and in the presence of a free upper surface of the upper layer. In this latter case, certainly not all possible wave motions

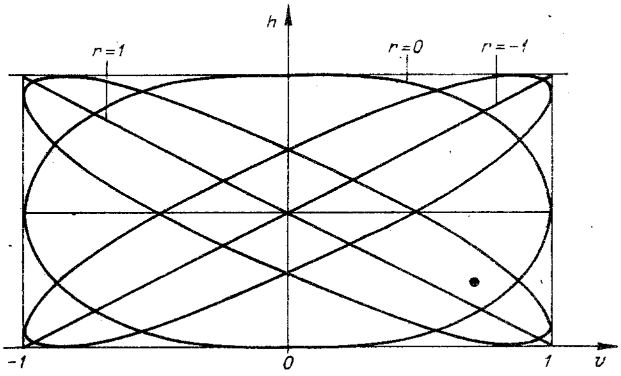

Fig. 5

will be encompassed by this description, but only those for which both relative Froude numbers $F_i = |u_2 - u_1|/\sqrt{h_i}$ (i = 1, 2) are on the order of $\sqrt{\mu}$. In other words, this is motion with relatively low slip velocities of the upper layer on the lower layer.

QUALITATIVE PROPERTIES OF MODEL III

The plane of the variables (v, h) will henceforth be called the "hodograph plane" (this terminology is used below without the quotation marks). In the sense of the derivation, (14) should be considered in the strip $0 \leq h \leq 1$.

The equations for the characteristics of the system (14)

$$dx/dt = k_\pm = (1-2h)v \pm \sqrt{h(1-h)}\sqrt{1-v^2} \quad (17)$$

show that this system is hyperbolic for $|v| < 1$ and elliptic for $|v| > 1$. The Riemann invariants, i.e., those functions $r = r(v, h)$ for which $r_t + kr_x = 0$ because of (14), are found explicitly here

$$r_\pm = (1-2h)v \pm 2\sqrt{h(1-h)}\sqrt{1-v^2}, \quad (18)$$

where both upper or both lower signs should be taken simultaneously in (17) and (18). Since the characteristics are level lines of the Riemann invariants, then the lines $r_\pm$ = const are the image of the characteristics on the hodograph plane. All these lines are contained in the rectangle of hyperbolicity $|v| \leq 1$, $0 \leq h \leq 1$. Moreover, (18) for $r = r_\pm$ is equivalent to the equation

$$(v + r(2h-1))^2 + (1-r^2)(2h-1)^2 = 1 - r^2.$$

Hence, $|r| \leq 1$, where the images of the characteristics are ellipses inscribed in the rectangle of hyperbolicity (Fig. 5). Now it is seen what is the above-mentioned difference in the behavior of the characteristics for models II and III: At the same time as the characteristics have a reentry point on the line of parabolicity in the model II, they are tangent to an analogous line in model III.

It is useful to note that the relationships

$$4k_+ = 3r_+ + r_-,\ 4k_- = r_+ + 3r_- \quad (19)$$

follow from (17) and (18). Hence, because of the inequalities $|r_\pm| \leq 1$, the following estimates are obtained:

$$3r_\pm - 1 \leqslant k_\pm \leqslant 3r_\pm + 1.$$

By using the Riemann invariants the simple waves are described in elementary terms, as particular solutions of the system (14) for which a functional relation of the form $F(v, h) = 0$ is valid. As is known, such a relation has the form of constancy of one of the Riemann invariants. The Riemann invariant r_+ is constant identically in a simple r_+ wave, and since the invariant r_- is also constant along each characteristic with the angular coefficient k_-, then these characteristics are straight lines on the (x, t) plane. In particular, the equation of the families of rectilinear characteristics in a simple r_+ wave centered at the point (0, 0) has the form

$$x = (1/4)(r_+ + 3r_-)t \quad (-1 \leqslant r_- \leqslant 1).$$

The places of r_+ and r_- must be interchanged for a r_- wave. Seeking the functions v and h in a simple wave is simplified if new quantities φ and θ are introduced by means of the formulas

$$v = \sin\varphi,\ h = \sin^2(\theta/2) \quad (-\pi/2 \leqslant \varphi \leqslant \pi/2,\ 0 \leqslant \theta \leqslant \pi). \quad (20)$$

It follows from (18) that with these quantities

$$r_+ = \sin(\varphi + \theta),\ r_- = \sin(\varphi - \theta).$$

Hence, if we put $\xi = x/t$, then for the centered simple r_+ wave, for instance, $3r_- = 4\xi - r_+$ and

$$\varphi + \theta = \arcsin r_+, \quad \varphi - \theta = \arcsin ((4\xi - r_+)/3),$$

from which φ and θ are found, and then v and h by means of (20).

EQUATIONS ON THE HODOGRAPH PLANE

As all quasilinear homogeneous autonomous systems of two equations in two independent variables, the system (14) is linearized by transfer to the hodograph plane, i.e., by that transformation of variables in which v and h become independent variables. For example, an examination of the quantities (x, t) in the system (14) as functions of (v, h) results in the equations

$$x_v = (1 - 2h)vt_v - h(1 - h)t_h, \quad x_h = (1 - 2h)vt_h - (1 - v^2)t_v. \tag{21}$$

Hence, x is eliminated by cross differentiation, and one linear equation in $t = t(v, h)$ is obtained:

$$h(1 - h)t_{hh} - (1 - v^2)t_{vv} + 2(1 - 2h)t_h + 4vt_v = 0. \tag{22}$$

For convenience in the analytical investigation, it is expedient to reduce (22) to canonical form by going over to the characteristic variables (the Riemann invariants) $r = r_+$ and $s = r_-$. Such a transformation can be performed by two means: direct replacement of the variables in (22), or by duplicating the previous means partially by starting from the characteristic form of the system (14)

$$r_t + k_+ r_x = 0, \quad s_t + k_- s_x = 0,$$

equivalent to the system

$$x_s = k_+ t_s, \quad x_r = k_- t_r,$$

which is rewritten in the form

$$4x_s = (3r + s)t_s, \quad 4x_r = (r + 3s)t_r \tag{23}$$

when (19) is taken into account. Elimination of the function x from (23) results in the canonical Euler–Poisson equation

$$t_{rs} = \frac{3}{2}\frac{1}{r - s}(t_r - t_s). \tag{24}$$

The second canonical form of (22) [or (24)] is obtained as a result of the change of variables

$$\xi = (1 - 2h)v = \frac{r - s}{2}, \quad \eta = \frac{1}{h(1 - h)(1 - v^2)} = \frac{16}{(r - s)^{2}}, \quad t(r, s) = z(\xi, \eta)$$

and turns out to be the following:

$$\eta^3 z_{\eta\eta} - z_{\xi\xi} = 0. \tag{25}$$

Comparing (24) with the analogous equation which occurs in the gasdynamics of a polytropic gas in the description of isentropic motions for which the numerical coefficient equals $(\gamma + 1)/2(\gamma - 1)$ (γ is the polytropy index) shows that these equations agree when $\gamma = 2$. The analogy which it is desired to establish in exact form hence occurs. This turns out to be possible, namely, if the following new quantities are introduced:

$$\tilde{\rho} = h(1 - h)(1 - v^2), \quad \tilde{u} = (1 - 2h)v, \tag{26}$$

then the system (14) is converted into the equivalent system

$$\tilde{\rho}_t + \tilde{u}\tilde{\rho}_x + \tilde{\rho}\tilde{u}_x = 0, \quad \tilde{u}_t + \tilde{u}\tilde{u}_x + \tilde{\rho}_x = 0, \tag{27}$$

i.e., into exactly the system of gasdynamics equations for a polytropic gas with $\gamma = 2$ (and the equation of state $\tilde{p} = (1/2)\tilde{\rho}^2$).

The unexpected, and, in a known sense, paradoxical deduction follows: The gasdynamic equations (27) have a definite physical meaning for negative values of the gas density $\tilde{\rho}$!

The Jacobian of the transition $(v, h) \rightarrow (\tilde{u}, \tilde{\rho})$ is evaluated to clarify those singularities which occur upon substituting (26). It turns out to equal,

$$\Delta = \partial(\tilde{u}, \tilde{\rho})/\partial(v, h) = (1 - 2h)^2 - v^2,$$

and vanishes on the characteristics $r_\pm = 1$. Meanwhile, the Jacobian Δ changes sign when passing through the line $|v| = 1$.

CONCLUDING REMARKS

In connection with the singularities detected in the behavior of two-layer "shallow water," the question naturally arises about what correct boundary value problems are possible for the models considered here. There are definite difficulties in this question which occur because of the partial ellipticity of the resulting differential equations. Problems of the Dirichlet type should be characteristic for the elliptic case. However, formulation of Dirichlet problems on the plane of the independent variables (x, t), where t is the time, raises doubts from the viewpoint of its physical meaningfulness. Meanwhile, the physically natural Cauchy problem with initial data at $t=0$ for elliptical data can turn out to be mathematically incorrect. It is here necessary to proceed carefully since the equations are nonlinear and it is difficult to predict any definite result of interaction between stability and nonlinearity.

The study of a strong discontinuity, which is natural for data in the hyperbolic domain, can possibly assist in this question. Construction of generalized solutions with strong discontinuities requires the involvement of some conservation laws of the form (8). As has already been noted, the choice is strictly limited in model I, and if some known analogies follow, then it should be expedient to set the two mass conservation laws, valid for this model, as well as the total momentum and total energy conservation laws, as the basis. The strong discontinuity equations obtained are hence quite complex and difficult to analyze. Conversely, there is an unlimited reserve of linearly independent conservation laws in models II and III, and the result will depend to a significant extent on their successful selection. The unique solvability of the problem on the dissociation of an arbitrary discontinuity under any admissible initial state can hence be proposed as the criterion for success. This requirement is not trivial, as can be seen in the example of the gasdynamics equations (27) if negative values of the density are also considered allowable.

The elucidated one-dimensional models and the preliminary qualitative deductions obtained for them allow a natural extension in several areas. Among these, for instance, are models of wave motions of multilayer "shallow water," two dimensional ("planar") motions, flows over a rough bottom, etc. However, the significance of the corresponding analytical investigations will be determined in large measure by progress in studying the one dimensional problems.

LITERATURE CITED

1. J. J. Stoker, Water Waves, Wiley-Interscience (1957).
2. K. O. Friedrichs, "On the derivation of the shallow water theory," Appendix to "The formation of breakers and bores" by J. J. Stoker, Commun. Pure Appl. Math., 1, 81-85 (1948).
3. L. V. Ovsyannikov, "On the foundation of shallow water theory," in: Dynamics of a Continuous Medium [in Russian], No. 15, Hydrodynamics Inst. Siberian Branch, Academy of Sciences of the USSR, Novosibirsk (1973).
4. L. V. Ovsyannikov, "Foundation of shallow-water theory," in: Transactions of the All-Union Conf. on Partial Differential Equations devoted to the 75th Birthday of Academician I. G. Petrovskii [in Russian], Mosk. Gos. Univ., Moscow (1978).
5. H. Lamb, Hydrodynamics, Dover (1932).
6. A. A. Ivanov (editor), Internal Waves [Russian translation], Mir, Moscow (1964).
7. N. E. Kochin, "Exact determination of steady waves of finite amplitude on the interface of two fluids of finite depth," Collected Works [in Russian], Vol. 2, Moscow–Leningrad (1949), pp. 43-79.